§3.3 Complex roots of the Characteristic Equation

Our 2nd order DE

$$ay'' + by' + cy = 0$$

where a, b, c are constants. The characteristic equation is

$$ar^2 + br + c = 0$$

so

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What happens when $b^2 - 4ac < 0$?

We have roots r_1 and r_2 which are complex conjugates

$$r_1 = \lambda + i\mu \quad \text{and} \quad r_2 = \lambda - i\mu$$

where λ and μ are real numbers.

Our solutions are

$$y_1 = e^{(\lambda + i\mu)t} \quad \text{and} \quad y_2 = e^{(\lambda - i\mu)t}$$

What does e to a complex power mean?

Euler's Formula

Recall the Taylor Series for e^t about $t=0$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots$$

Now,

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$$

$$= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$= \cos(t) + i\sin(t)$$

Thus,

$$e^{it} = \cos(t) + i\sin(t)$$

and this is called Euler's formula.

Now,

$$y_1 = e^{(\lambda + i\mu)t}$$
$$= e^{\lambda t} e^{i\mu t}$$
$$= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$$

and

$$y_2 = e^{(\lambda - i\mu)t}$$
$$= e^{\lambda t} e^{-i\mu t}$$
$$= e^{\lambda t}(\cos(\mu t) - i\sin(\mu t))$$

Recall $\sin(-t) = -\sin(t)$ (odd function)
$\cos(-t) = \cos(t)$ (even function)

Complex roots: The general Solution

Recall that if $y = u + iv$ is a solution, then u and v are solutions. Then

$$u = e^{\lambda t}\cos(\mu t) \quad \text{and} \quad v = e^{\lambda t}\sin(\mu t)$$

are solutions. The Wronskian is

$$W[u,v] = \begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix}$$

$$= \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t)$$
$$- \lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t)$$

$$= \mu e^{2\lambda t}(\cos^2(\mu t) + \sin^2(\mu t))$$

$$= \mu e^{2\lambda t}.$$

So $W[u,v] \neq 0$ if $\mu \neq 0$. Thus, u, v form a fundamental set of solutions if $\mu \neq 0$.

Remark If $\mu = 0$, then $r_1 = \lambda$ and $r_2 = \lambda$
so we have a real repeated root.

Our general solution is

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$$

Ex $y'' + y' + 9.25\, y = 0$

$y(0) = 2, \quad y'(0) = 8$

The characteristic equation is

$$r^2 + r + 9.25 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 37}}{2}$$

$$= \frac{-1 \pm 6i}{2}$$

$$= \frac{-1}{2} \pm 3i .$$

Now, our general solution is

$$y = c_1 e^{-\frac{1}{2}t} \cos(3t) + c_2 e^{-\frac{1}{2}t} \sin(3t)$$

$$2 = y(0) = c_1$$

$$c_1 = 2$$

Now,

$$y = 2e^{-\frac{1}{2}t} \cos(3t) + c_2 e^{-\frac{1}{2}t} \sin(3t)$$

$$y' = -e^{-\frac{1}{2}t} \cos(3t) - 6e^{-\frac{1}{2}t} \sin(3t) - \frac{1}{2} c_2 e^{-\frac{1}{2}t} \sin(3t) + 3c_2 e^{-\frac{1}{2}t} \cos(3t)$$

$$8 = y'(0) = -1 + 3c_2$$

$$9 = 3c_2$$

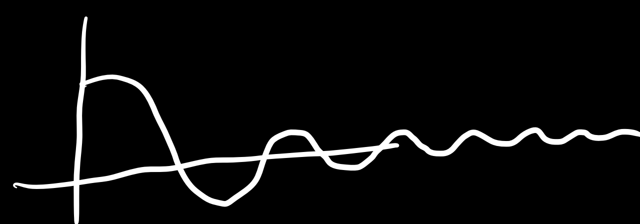
$$c_2 = 3$$

Thus,

$$y(t) = 2e^{-\frac{1}{2}t}\cos(3t) + 3e^{-\frac{1}{2}t}\sin(3t).$$

Notice that this is an oscillating solution with

$$\lim_{t\to\infty} y(t) = 0$$



§3.4 Repeated Roots and Reduction of Order

Our roots of the characteristic equation are

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

and we now consider the case where

$b^2-4ac=0$, so

$$r = \frac{-b}{2a}.$$

That is, we have one real repeated root.

We have the solution

$$y_1 = e^{\frac{-b}{2a}t}$$

What about the second solution?

We know that if y_1 is a solution,

then

$$y_2 = c\,y_1$$

is also a solution. Let's extend this idea by considering

$$y_2 = v(t) y_1$$

where $v(t)$ is a to-be-determined function. For y_2 to be a solution, it needs to satisfy the DE

$$ay'' + by' + cy = 0.$$

We have

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$$

$$= v'' y_1 + 2 v' y_1' + v y_1''$$

Now, plugging into the DE

$$a y_2'' + b y_2' + c y_2 = a\left(v'' y_1 + 2v' y_1' + v y_1''\right)$$

$$+ b(v'y_1 + vy_1') + cvy_1$$

$$= y_1(av'' + bv') + 2av'y_1'$$

$$+ v(ay_1'' + by_1' + cy_1)$$

$$= y_1(av'' + bv') + 2av'y_1'$$

Thus, we have the DE for v

$$y_1 a v'' + (by_1 + 2ay_1')v' = 0$$

Now, $y_1 = e^{\frac{-b}{2a}t}$, so

$$e^{\frac{-b}{2a}t} a v'' + \left(b e^{\frac{-b}{2a}t} + 2a\left(\frac{-b}{2a}\right)e^{\frac{-b}{2a}t}\right)v' = 0$$

$$e^{\frac{-b}{2a}t} a v'' = 0$$

Since $e^{\frac{-b}{2a}} \neq 0$, we must have that

$v'' = 0$. Thus, $v(t) = c_1 + c_2 t$.

Now, we have

$$y_1 = e^{\frac{-b}{2a}t} \quad \text{and} \quad y_2 = (c_1 + c_2 t)\, e^{\frac{-b}{2a}t}$$

so a linear combination of y_1 and y_2 yields

$$\begin{aligned} y &= c_3 e^{\frac{-b}{2a}t} + c_4 [c_1 + c_2 t)\, e^{\frac{-b}{2a}t} \\ &= (c_3 + c_1 c_4)\, e^{\frac{-b}{2a}t} + c_2 c_4 t\, e^{\frac{-b}{2a}t} \\ &= \tilde{c}_1 e^{\frac{-b}{2a}t} + \tilde{c}_2 t\, e^{\frac{-b}{2a}t} \end{aligned}$$

i.e.

$$y_1 = e^{\frac{-b}{2a}t} \quad \text{and} \quad y_2 = t\, e^{\frac{-b}{2a}t}$$

Checking the Wronskian,

$$W[y_1, y_2] = \begin{vmatrix} e^{\frac{-b}{2a}t} & t\, e^{\frac{-b}{2a}t} \\ \frac{-b}{2a} e^{\frac{-b}{2a}t} & e^{\frac{-b}{2a}t} + \frac{-bt}{2a} e^{\frac{-b}{2a}t} \end{vmatrix}$$

$$= e^{-\frac{b}{a}t} - \frac{bt}{2a} e^{-\frac{b}{a}t} + \frac{bt}{2a} e^{-\frac{b}{a}t}$$

$$= e^{-\frac{b}{a}t}$$

We see that $W[y_1, y_2] \neq 0$. Therefore, y_1 and y_2 form a fundamental set of solution. Hence our general solution is

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}.$$

Ex $y'' - y' + \frac{y}{4} = 0$, $y(0) = 2$, $y'(0) = \frac{1}{3}$

Our characteristic equation is

$$r^2 - r + \frac{1}{4} = 0$$

$$(r - \tfrac{1}{2})^2 = 0$$

$$r = \tfrac{1}{2}.$$

Therefore, the general solution is

$$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

$$2 = y(0) = c_1$$

$$c_1 = 2$$

Now,

$$y = 2e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

$$y' = e^{\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} + \frac{1}{2} c_2 t e^{\frac{1}{2}t}$$

$$\frac{1}{3} = y'(0) = 1 + c_2$$

$$c_2 = \frac{-2}{3}.$$

Thus,

$$y(t) = 2e^{\frac{1}{2}t} - \frac{2}{3} t e^{\frac{1}{2}t}.$$

Reduction of Order

Consider the 2^{nd}-order DE

$$y'' + p(t)y' + q(t)y = 0$$

and suppose we know $y_1(t)$ is one of the solutions. To find a second solution, let

$$y_2 = v(t)y_1(t).$$

Then

$$y_2' = v'y_1 + vy_1'$$

and

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$
$$= v''y_1 + 2v'y_1' + vy_1''$$

Now, plugging into the DE, we get

$$v''y_1 + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + qvy_1$$

$$= y_1 v'' + (2y_1' + py_1)v' + v(y_1'' + py_1' + qy_1)$$

$$= y_1 v'' + (2y_1' + py_1)v'$$

We get the DE for v

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Note that y_1 is known. The 2nd-order DE for v is really a 1st-order DE for v'. That is, Let $w = v'$, so

$$y_1 w' + (2y_1' + py_1)w = 0$$

This DE is separable. Once we find w, we can find v by integrating w.

<u>Ex</u> Suppose $y_1 = t^{-1}$ is solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0$$

Find a fundamental set of solutions.

Let $\quad y_2 = v(t)t^{-1}$

So,

$$y_2' = v't^{-1} - vt^{-2}$$

$$y_2'' = v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3}$$

$$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$$

Now, plugging in to the DE,

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} = 0$$

$$2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$

$$2v''t - v' = 0$$

Let $\quad w = v'$. Then

$$2w't - w = 0$$

$$w' = \frac{w}{2t}$$

$$\frac{1}{w} w' = \frac{1}{2t}$$

$$\int \frac{d}{dt} (\ln w) dt = \int \frac{1}{2t} dt$$

$$\ln w = \frac{1}{2} \ln t + C$$

$$w = C e^{\frac{1}{2}\ln t}$$

$$w = C e^{\ln t^{1/2}}$$

$$w = C t^{1/2}$$

Now,

$$w = v'$$

so

$$v' = C t^{\frac{1}{2}}$$

$$\int v'\,dt = \int c\,t^{\frac{1}{2}}\,dt$$

$$v(t) = \frac{2c}{3}t^{3/2} + C_2$$

Then

$$\begin{aligned} y_2 &= v\,y_1 \\ &= \left(\frac{2}{3}c\,t^{\frac{3}{2}} + C_2\right)t^{-1} \\ &= \frac{2}{3}c\,t^{\frac{1}{2}} + C_2\,t^{-1} \end{aligned}$$

Our general solution is

$$\begin{aligned} y &= C_3\,y_1 + C_4\,y_2 \\ &= C_3\,t^{-1} + C_4\left(\frac{2}{3}c\,t^{\frac{1}{2}} + C_2\,t^{-1}\right) \\ &= (C_3 + C_2C_4)\,t^{-1} + \frac{2}{3}c\,C_4\,t^{\frac{1}{2}} \\ &= \tilde{C}_1\,t^{-1} + \tilde{C}_2\,t^{\frac{1}{2}}. \end{aligned}$$